Parametric Resonance and Theory of Bragg Waveguides

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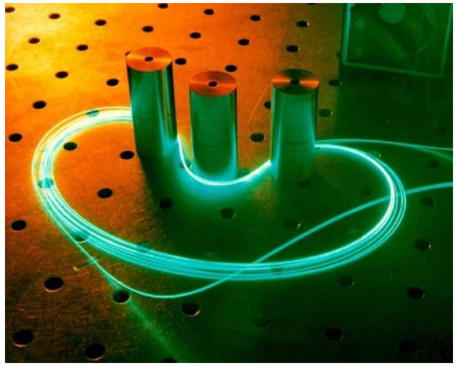
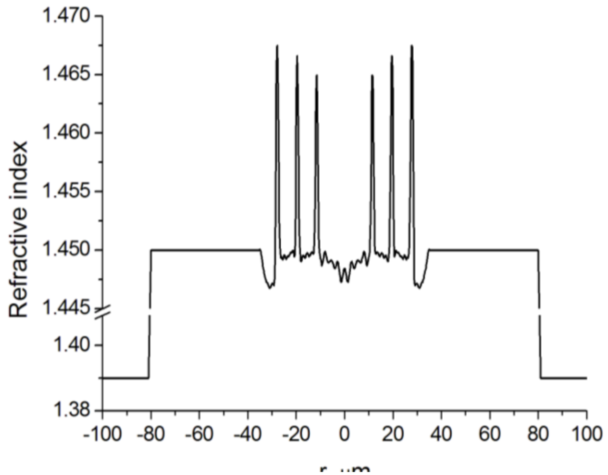
Introduction

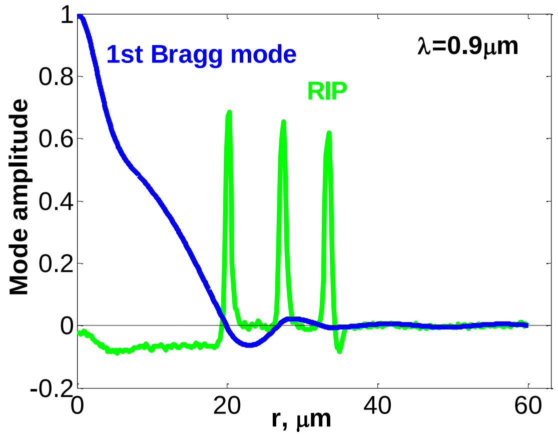
During the last decades, powerful infrared lasers made of Iterbium-doped silica fibers are actively designed and tested.

The main restriction upon the radiation power is imposed by the quartz destruction threshold of about 1 MW/cm^2. Therefore a key design element is the fiber diameter increase, however not violating the single-mode generation regime.

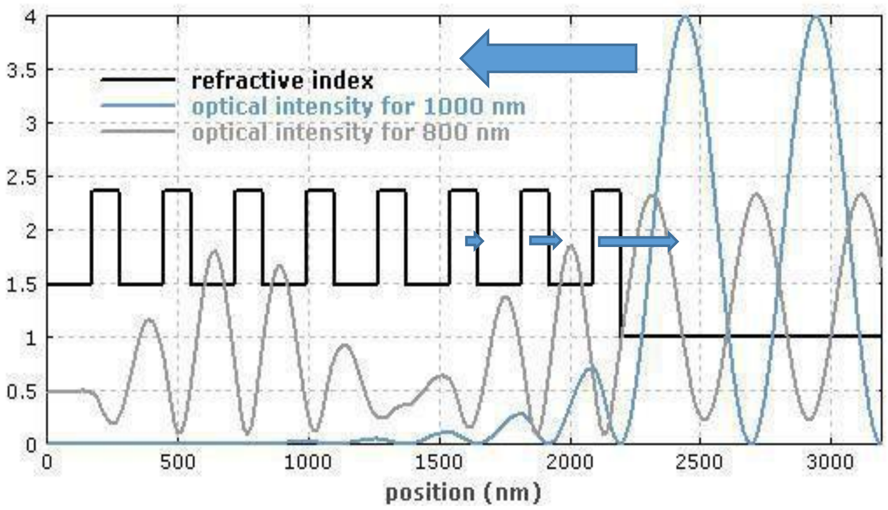
One of the methods of energy localization in the optical fiber core consists in quasi-periodic modulation of the refraction index in its cladding due to adding a few layers doped with germanium dioxide.
A specially designed structure of circular doped layers assures resonance field attenuation in the fiber cladding, impeding energy leakage from the guide core. This effect, analogous to Bragg reflection in crystal gratings, gives way to creation of high-quality optical fibers with large mode area – P. Roy, S. Février, et al. (XLIM Research Institute, Limoges, France), M. Likhachev, S. Semjonov (Fiber Optics Research Center, Moscow).

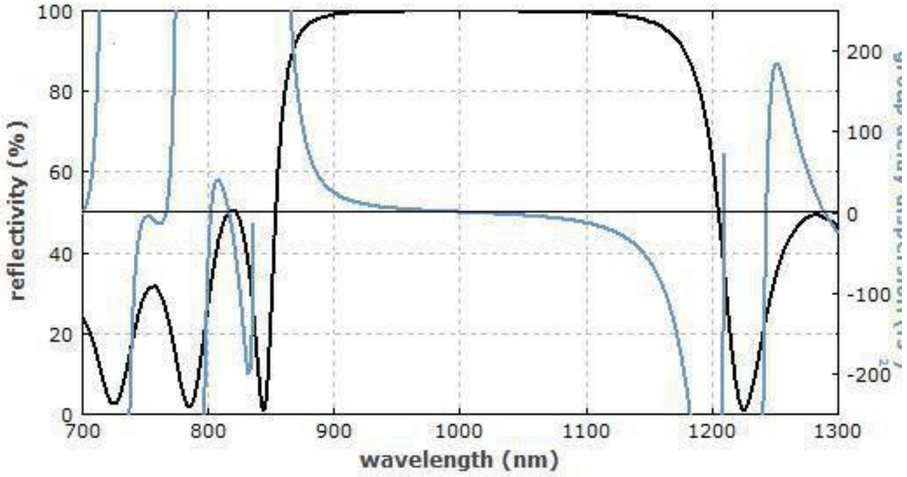






Introduction





Elementary theory of Bragg mirrors is well known. The refractive index and thickness of alternating dielectric layers is designed in such a way that the Fresnel reflections of the incident wave from their interfaces add up constructively (in-phase). This leads to an exponential field attenuation in the semi-infinite periodic mirror structure, hence the total reflection of the incident wave.

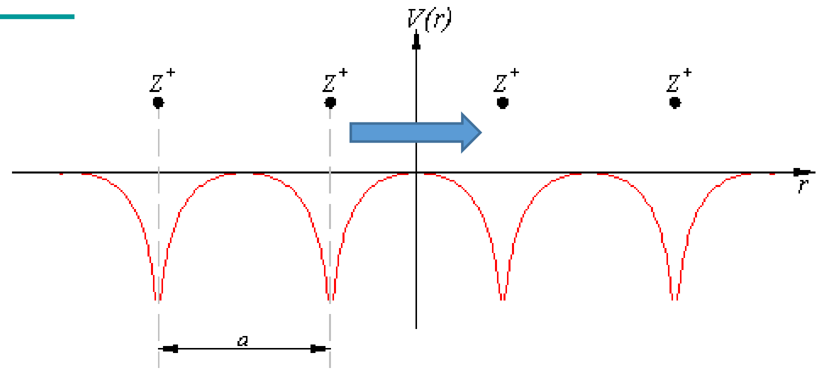
As the phase advance depends on frequency, the field attenuation decrement and the reflection coefficient drop with changing wavelength, and, outside a certain frequency band, exponential attenuation gives place to an oscillatory regime.

When designing a Bragg waveguide cladding, the problem arises of providing maximum field attenuation in the periodic dielectric structure with minimum number of layers.

Introduction

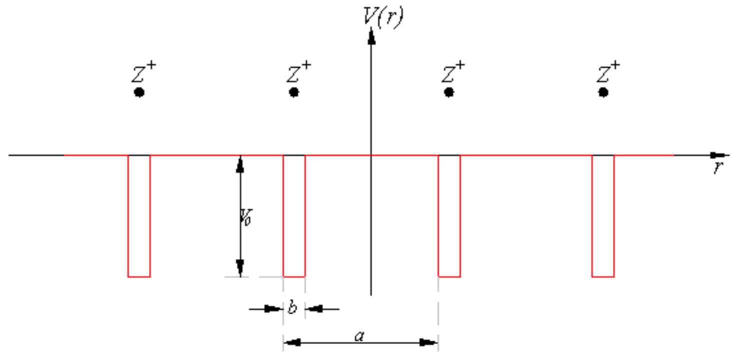
In order to find a solution of the aforementioned optimization problem, consider general theory of EM wave propagation in periodic media. It is analogous to the theory of Bloch waves in a periodic potential:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+V_a(x)\psi(x)=E\psi(x) \qquad V_a(x)=V_a(x+a).$$



1D propagation of electromagnetic waves in a non-uniform medium with refractive index $n(x)$ is governed by a similar equation:

$$E(x,t)=u(x)\exp(-i\omega t), \qquad \boxed{u''+k^2n^2(x)u=0, \quad k=\omega/c}$$



Analytic solutions of 1D Schrödinger or wave equation are known but for a few model potentials. Among periodic $n(x+\Lambda)=n(x)$, an exact solution can be easily constructed for a meander profile of refractive index or for a grid of Dirac delta-functions (A. Yariv, P. Yeh. Optical Waves in Crystals). In other cases one has to rely upon approximate formulas or numerical approaches. However, Floquet theorem allows us to predict the general character of the solution:

$$\boxed{\begin{aligned}&u(x)=e^{\mu x}P(x),\\ &P(x+\Lambda)=P(x)\end{aligned}}$$

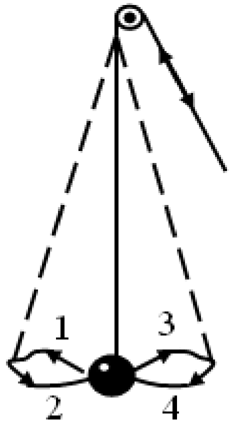
Introduction

In accordance with Floquet theorem, the product of the multipliers $\rho_{1,2} = \exp(\mu_{1,2}\Lambda)$ of two fundamental solutions equals to unity. Two situations are possible:

1. The multipliers are real-valued; one solution is exponentially growing and the other is proportionally decreasing: $\rho_2 = 1/\rho_1$
2. The multipliers are purely imaginary; they are complex conjugated and equal to unity by modulus.

In order to construct an interference mirror or a Bragg fiber cladding, one has to create an exponentially decreasing solution.
To comprehend the principle of searching an optimal design, consider the electro-mechanical analogy.
To a change of variables, one-dimentional wave equation with non-uniform $n(x)$
С точностью до замены переменных одномерное волновое уравнение с is equivalent to a linear oscillator with variable eigenfrequency $\omega(t)$:

$$u'' + q^2(x)u = 0, \quad q(x) = kn(x) \quad \Longleftrightarrow \quad \boxed{u'' + \omega^2(t)u = 0}$$



$$\omega(t) = \sqrt{\frac{g}{l(t)}}$$

In the case of periodic eigenfrequency variations: $\omega(t+T) = \omega(t)$
an exponential increase of the oscillation amplitude – parametric resonance may arise.

Introduction

Elementary theory of parametric resonance (L. Landau, E. Lifshitz. Mechanics) states that, with a small perturbation of the oscillator parameters by a harmonic function: $\omega^2(t) = \omega_0^2(1 + h\cos\gamma t)$, $\quad h << 1$, the resonance is excited most intensively if the modulation frequency is close to the double eigenfrequency ω_0 : $\quad \gamma = 2\omega_0 + \delta\omega$.

The solution to $\boxed{u'' + \omega_0^2(1 + h\cos\gamma t)u = 0}$ is sought as $\quad u = A(t)\cos(\omega_0 + \dfrac{\delta\omega}{2})t + B(t)\sin(\omega_0 + \dfrac{\delta\omega}{2})t$

The amplitudes $A(t), B(t)$, to the first-order approximation relative to $\delta\omega$, can be found from the equation set

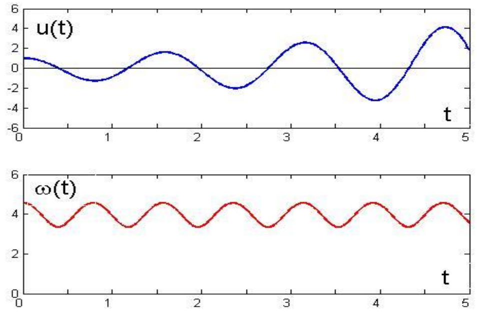
$$\begin{cases} 2A' + \left(\dfrac{h\omega_0}{2} + \delta\omega\right)B = 0, \\ 2B' + \left(\dfrac{h\omega_0}{2} - \delta\omega\right)A = 0 \end{cases}$$

having a solution proportional to $\exp(st)$, where $\boxed{s = \dfrac{1}{2}\sqrt{\left(\dfrac{h\omega_0}{2}\right)^2 - (\delta\omega_0)^2}}$

Maximum amplification on a period $T = \dfrac{2\pi}{\omega_0}$ equals to $\exp(\dfrac{\pi}{2}h)$

Frequency band where the resonance arises is determined by inequality $|\varepsilon| < \dfrac{h\omega_0}{2}$

With another choice of initial phase, a decreasing solution arises $\sim \exp(-st)$.



Introduction

Beyond perturbation theory, no elementary theory exists describing quasi-periodic solutions $u = \exp(\mu t) P(t)$. For the classical Mathieu equation $\boxed{u'' + \omega_0^2 \left(1 + h\cos\gamma t\right) u = 0}$ numerous algorithms of characteristic exponent μ and periodic function $P(t)$ determination have been developed.

However, these results have a limited application area, as in the case of deep modulation the oscillator parameters use to vary in a more complicated way.

The inverse problem – search for an optimal swinging regime is not enough theoretically studied.

Meanwhile we have a striking example of efficient parametric excitation of mechanical oscillations.

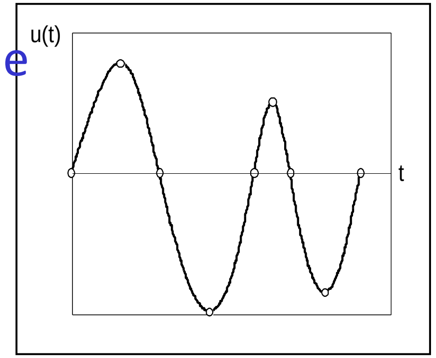
Contemplating a child rocking a swing we can make a number of useful observations:

- The child intuitively finds a law of changing the pendulum parameters leading to the fastest growing of the oscillation amplitude;
- He/she does not use a wristwatch but coordinates his/her movements with the current oscillation phase;
- Pendulum parameters vary periodically but far from harmonic way.

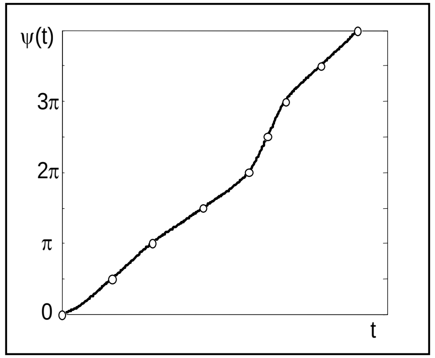
Action plan

In order to easily solve the inverse problem, it is desirable to have an analytical description of anharmonic linear oscillations with arbitrary varying eigenfrequency $\omega(t)$:

$$\boxed{u'' + \omega^2(t)u = 0}$$



amplitude



phase

Variations of eigenfrequency result in the change of the oscillation period, amplitude and waveform.

Basing on our experience in rocking the swing, let us use the phase of anharmonic oscillations as a new independent variable.

For a periodic $\omega(t)$, we will derive a continuum of exact solutions to the oscillation equation and find a simple formula for the Floquet characteristic number.

In what follows, we describe the basics of the phase parameter method, some generalizations and application to wave propagation in periodic media.

Linear oscillator: method of phase parameter

Let us define the phase of anharmonic oscillations by the following formula $\psi(t) = \operatorname{arcctg}\dfrac{u'(t)}{\omega(t)u(t)}$

and consider it as a new independent variable. We make a substitution $\omega(t) = \Omega(\psi)$

and look for a parametric solution $u = U(\psi), \quad t = T(\psi)$.

Oscillation equation $\boxed{u'' + \omega^2(t)u = 0}$ transforms to a set of nonlinear differential equations

$$\boxed{\dot{T}(\psi) = \frac{1}{\Omega(\psi)} - \frac{\dot{\Omega}(\psi)}{2\Omega^2(\psi)}\sin 2\psi, \quad \frac{\dot{U}(\psi)}{U(\psi)} = \operatorname{ctg}\psi - \frac{\dot{\Omega}(\psi)}{\Omega(\psi)}\cos^2\psi}$$

having an exact solution for an arbitrary function $\Omega(\psi)$:

$$\begin{cases} T(\psi) = \displaystyle\int\left(\frac{1}{\Omega} - \frac{\dot{\Omega}}{2\Omega^2}\sin 2\psi\right)d\psi = t_0 + \frac{1}{\bar{\omega}}\int e^{g(\psi)}\left[1 + \frac{1}{2}\dot{g}(\psi)\sin 2\psi\right]d\psi \\ U(\psi) = \dfrac{1}{\bar{\omega}}\sin\psi\,\exp[g(\psi)\cos^2\psi + \displaystyle\int_0^{\psi} g(\varphi)\sin 2\varphi\, d\varphi] \end{cases}$$

Here, for convenience, we denote:

$$\boxed{\Omega(\psi) \equiv \bar{\omega}\, e^{-g(\psi)}}$$

Periodic modulation

So, we obtain for *u(t)* an explicit analytical solution $t \equiv T(\psi), \quad u(t) \equiv U(\psi)$

with arbitrary dependence of the eigenfrequency on the oscillation phase: $\Omega(\psi) \equiv \varpi \exp[-g(\psi)]$

$$\begin{cases} T(\psi) = t_0 + \dfrac{1}{\varpi} \displaystyle\int e^{g(\psi)} \left[1 + \frac{1}{2} \dot{g}(\psi) \sin 2\psi \right] d\psi \\ U(\psi) = \dfrac{1}{\varpi} \sin\psi \exp[g(\psi)\cos^2\psi + \displaystyle\int_0^{\psi} g(\varphi) \sin 2\varphi \, d\varphi] \end{cases}$$

For a periodic variation of the parameter: $g(\psi + \pi) = g(\psi)$, the integrals contain linearly growing "secular" terms

Coming back to time variable *t* , we obtain Floquet solution $\boxed{u(t) = \tilde{u}(t)\, e^{\frac{\nu}{\tau} t}, \qquad \tilde{u}(t + 2\tau) = \tilde{u}(t)}$

and explicit formulas for its quasi-period and increment:

$$\boxed{\tau = \frac{2}{\varpi} \int_0^{\pi} e^{g(\psi)} \sin^2\psi \, d\psi} \qquad \boxed{\nu = \int_0^{\pi} g(\psi) \sin 2\psi \, d\psi}$$

Parametric resonance

$$\omega(t) \equiv \Omega(\psi) = \omega_0 \exp[-g(\psi)]$$

$$g(\psi + 2\pi) = g(\psi)$$

The method of phase parameter reveals a clear relation between characteristic Floquet exponent and the modulation law of the oscillator parameters.

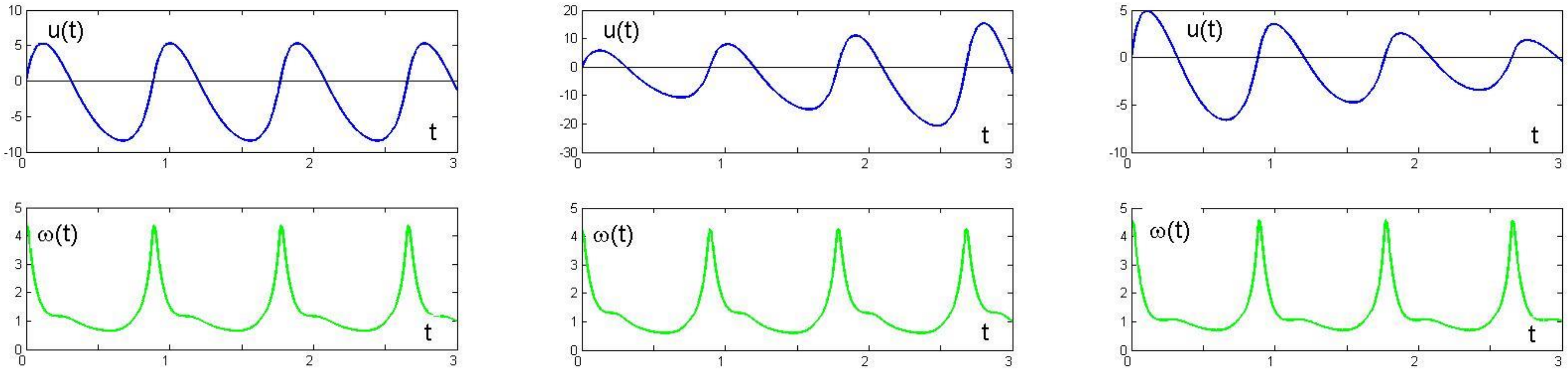
Expanding $g(\psi)$ in a Fourier series we obtain:

$$g(\psi) = a_0 + \sum_{m=1}^{\infty} (a_m \cos m\psi + b_m \sin m\psi)$$

$$\boxed{\nu = \int_0^{\pi} g(\psi) \sin 2\psi \, d\psi = \frac{\pi}{2} b_2}$$

Only second odd harmonic is important!

Numerical example: $\quad g(\psi) = \sin\psi + 2\cos\psi + \cos 2\psi + b_2 \sin 2\psi$



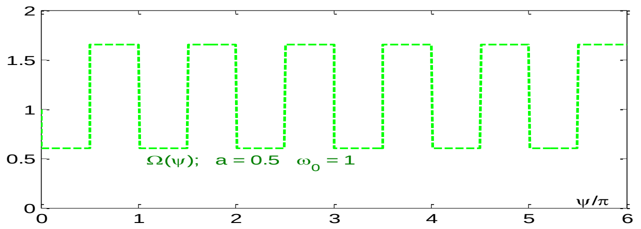
$b_2 = 0 \qquad\qquad b_2 = 0.3 \qquad\qquad b_2 = -0.3$

Inverse problem – maximum increment

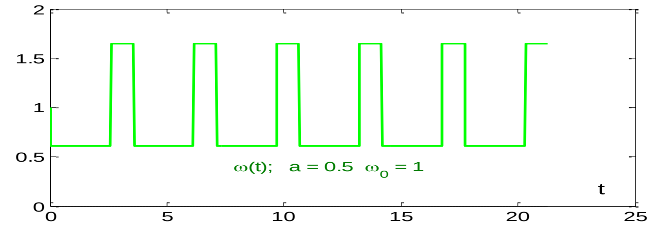
Obvious inequality: $$\nu = \int_0^{\pi} g(\psi)\sin 2\psi\, d\psi < \left\{ \max_{0<\psi<\pi/2}[g(\psi)] - \min_{\pi/2<\psi<\pi}[g(\psi)] \right\} \underbrace{\int_0^{\pi/2} \sin 2\psi\, d\psi}_{=1}$$

shows that, within given eigenfrequency variation limits, maximum oscillation increment is reached for a step-wise parameter change at each quarter of period:

$$\Omega(\psi) = \omega_0 \exp[-a\,\mathrm{sign}(\sin 2\psi)]$$

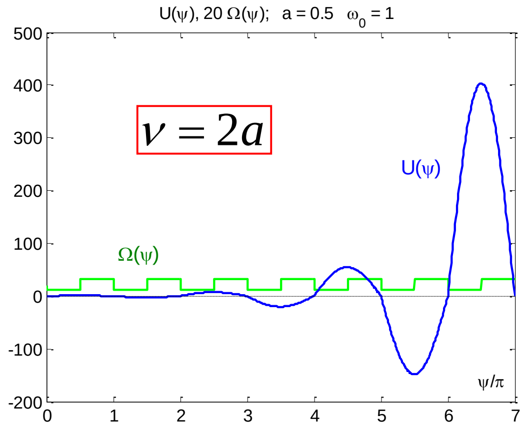


$$\omega(t) = \Omega[\Psi(t)]$$

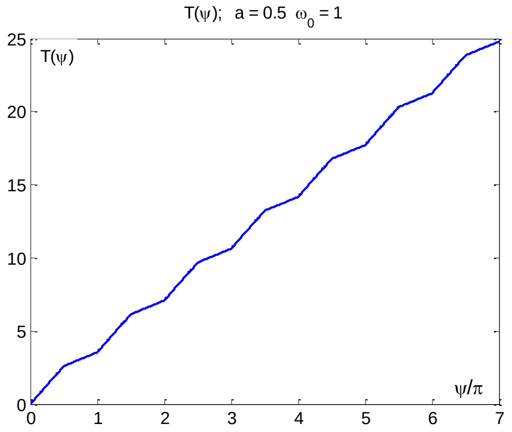


Parametric solution:

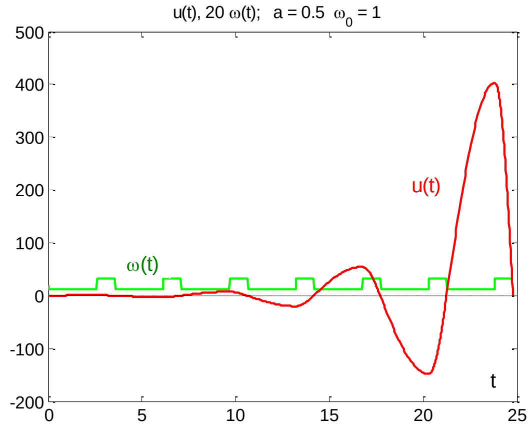
In physical variables:



$U(\psi)$



$T(\psi)$



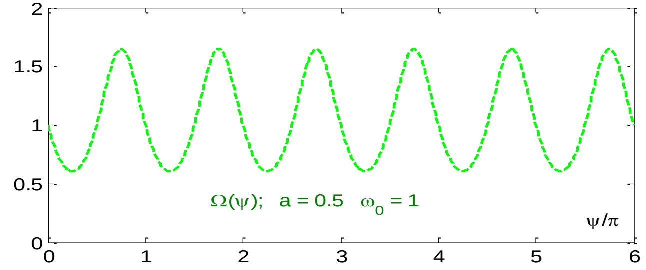
$u(t)$

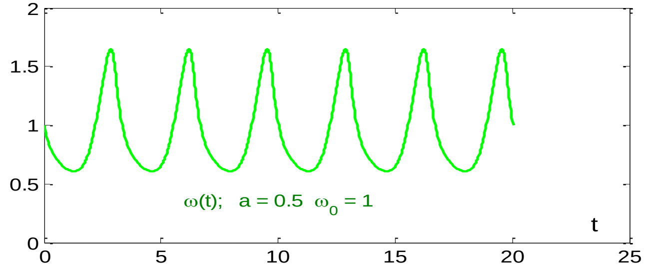
Optimal solution

In real life, the step-wise parameter change is hardly realizable. An optimal smooth solution corresponds to a single harmonic in the exponent $\Omega(\psi) = \omega_0 \exp[-g(\psi)]$

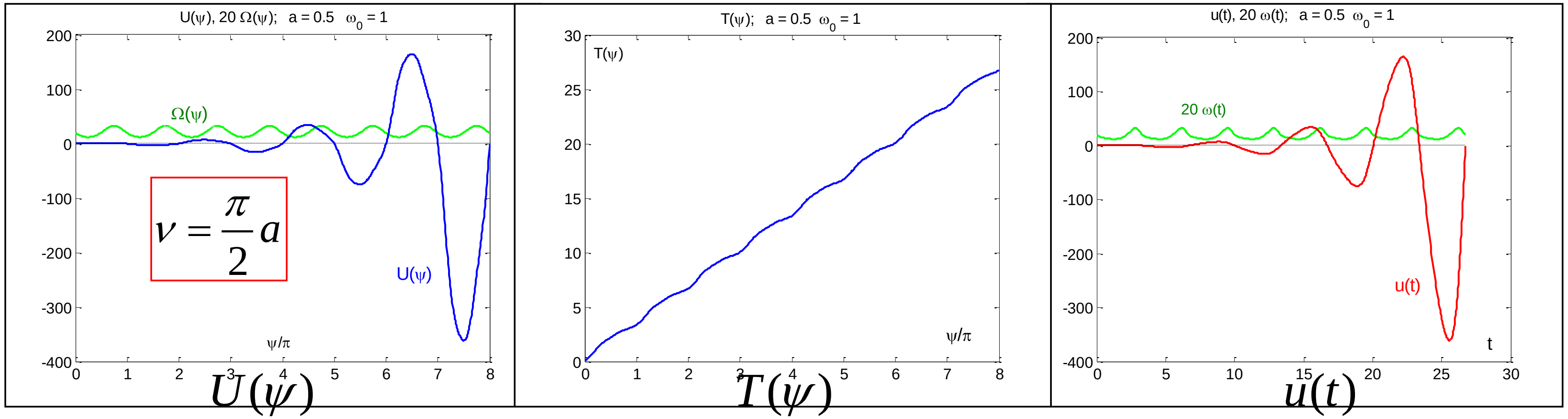
$$\boxed{\Omega(\psi) = \omega_0 \exp(-a \sin 2\psi)}$$

$$\omega(t) = \Omega[\Psi(t)]$$





$$\boxed{U(\psi) = A \sin\psi \exp\left[\frac{a}{2}\left(\psi + \sin 2\psi + \frac{1}{4}\sin 4\psi\right)\right], \quad T(\psi) = \frac{1}{\omega_0}\int_0^{\psi} e^{a \sin 2\psi}\left(1 + \frac{a}{2}\sin 4\psi\right) d\psi}$$



$$\boxed{\nu = \frac{\pi}{2} a}$$

$U(\psi)$ $T(\psi)$ $u(t)$

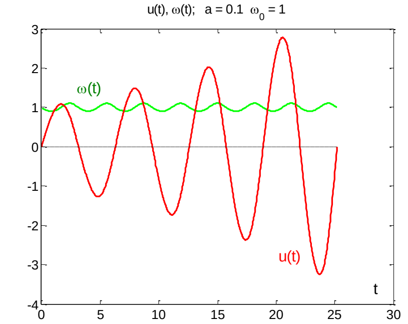
Intermediate summary

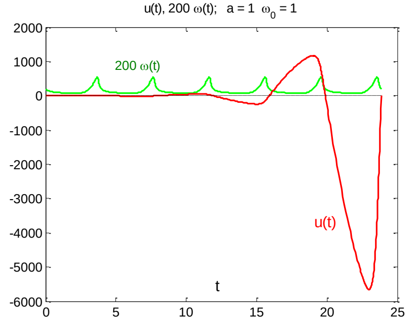
$$\psi(t) = \operatorname{arcctg} \frac{u'(t)}{\omega(t)u(t)}$$

$$u'' + \omega^2(t)u = 0$$

$$\nu \sim \int_0^{\pi} \ln \Omega(\psi) \sin 2\psi \, d\psi$$

- An explicit definition of phase of linear anharmonic oscillations is given.
- Using the oscillation phase as independent variable yields an exact analytic solution for arbitrary dependence $\omega(t) = \Omega(\psi)$.
- Explicit formula is found for the increment of parametric oscillations:
- For a weak periodic modulation, our solution is in agreement with classical theory of parametric resonance
- In case of deep modulation, we find an optimal regime of resonance excitation





Nonlinear oscillations

Method of phase parameter can be applied to describe nonlinear oscillations as well.

Consider physical pendulum motion:

$$\boxed{u'' + \omega^2(t)\sin u = 0}$$

We define oscillarion phase as and look for a parametric solution:

$$\boxed{\operatorname{ctg}\psi = \frac{u'}{\omega\, p(u)}, \quad p(u) = 2\sin\frac{u}{2}}$$

$$t = T(\psi), \quad \omega[T(\psi)] = \Omega(\psi), \; u[T(\psi)] = U(\psi)$$

$$\boxed{\begin{aligned}\dot T(\psi) &= \frac{\Omega - \dot\Omega \sin\psi\cos\psi}{\Omega^2 \cos U/2} \\ \dot U(\psi) &= 2\operatorname{tg}\frac{U}{2}\,\left(\operatorname{ctg}\psi - \frac{\dot\Omega}{\Omega}\cos^2\psi\right)\end{aligned}}$$

These equations have analytic solution for arbitrary

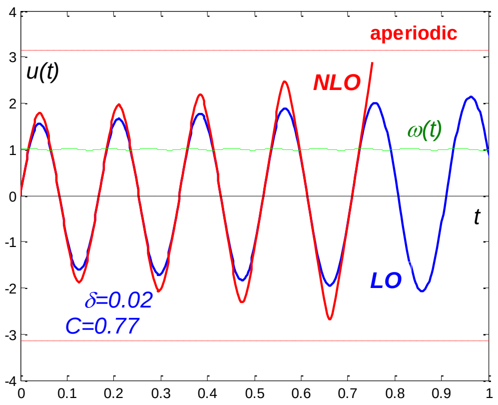
$$\boxed{\Omega(\psi) = \bar\omega \exp[-g(\psi)]}$$

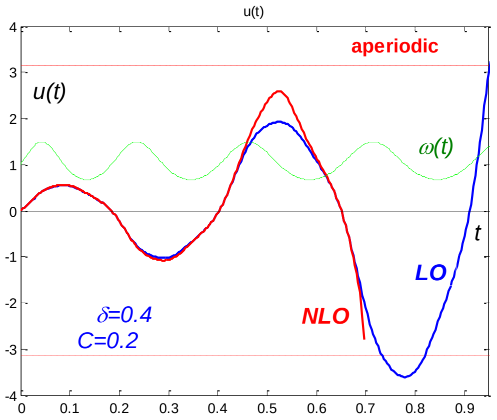
$$\boxed{\begin{aligned}\sin\frac{U(\psi)}{2} &= C\sin\psi \exp\left[\int_0^{\psi} \dot g(\varphi)\cos^2\varphi\, d\varphi\right], \\ T(\psi) &= t_0 + \frac{1}{\bar\omega}\int_0^{\psi} \exp[g(\varphi)]\left[1 + \frac{1}{2}\dot g(\varphi)\sin 2\varphi\right]\frac{d\varphi}{\cos\dfrac{U(\psi)}{2}}\end{aligned}}$$

Nonlinear Floquet theorem:

$$\boxed{\begin{aligned}\sin\frac{U(\psi+\pi)}{2} &= -\sin\frac{U(\psi)}{2}\cdot e^{\nu}, \\ \nu &= \int_0^{\pi} g(\varphi)\sin 2\varphi\, d\varphi\end{aligned}}$$

Parametric resonance of linear and nonlinear oscillator:





Waves in periodic media

The results obtained for a linear oscillator can be applied to 1D wave equation by change of variables

$$E(x,t) = u(x)\exp(-i\omega t)$$

$$u'' + q^2(x)u = 0, \quad q(x) = kn(x), \quad k = \omega/c$$

$$t \to x, \;\; \omega(t) \to q(x)$$

Phase parameter:

$$\frac{u'(x)}{q(x)u(x)} = \operatorname{ctg}\psi(x), \quad q(x) = Q(\psi)$$

$$x \equiv X(\psi), \quad u \equiv U(\psi)$$

Nonlinear parametric equations

$$\dot{X}(\psi) = \frac{1}{Q(\psi)} - \frac{\dot{Q}(\psi)}{2Q^2(\psi)}\sin 2\psi$$

$$\frac{\dot{U}(\psi)}{U(\psi)} = \frac{u'}{u}\cdot\dot{X} = \operatorname{ctg}\psi - \frac{\dot{Q}(\psi)}{Q(\psi)}\cos^2\psi$$

Have exact analytic solution for arbitrary $Q(\psi)$

For periodic $q(x) = Q(\psi)$ we obtain analytic description of Bloch waves in a “forbidden” zone

Problems of optimal synthesis

1. Multilayer mirror

The design goal – provideь maximum decrement (attenuation on a period)

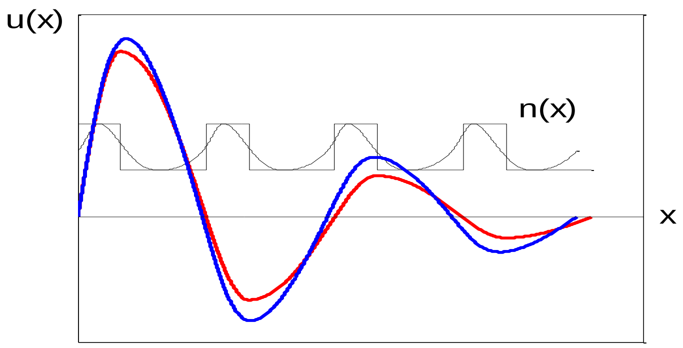
$$\nu = \int_0^{\pi} g(\psi)\sin 2\psi \, d\psi$$ within technological limits: $\boxed{n_2 < n(x) < n_1}$

Solution: $n(x) = \bar{n}\exp[g(\psi)]$

a) $g(\psi) = \begin{cases} \ln(n_1), & 0<\psi<\pi/2 \\ \ln(n_2), & \pi/2<\psi<\pi \end{cases}$ - ideal meander function - (absolute maximum: $\nu = \ln\frac{n_1}{n_2}$)

b) $g(\psi) = \frac{1}{2}\ln\left(\frac{n_1}{n_2}\right)\sin 2\psi$ - optimal smooth profile, $\nu = \frac{\pi}{4}\ln\frac{n_1}{n_2}$

Coming back to x variable we obtain a stack of quarter-period plates and its smooth analog:

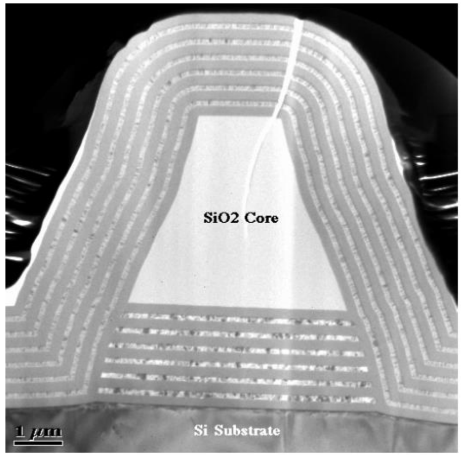


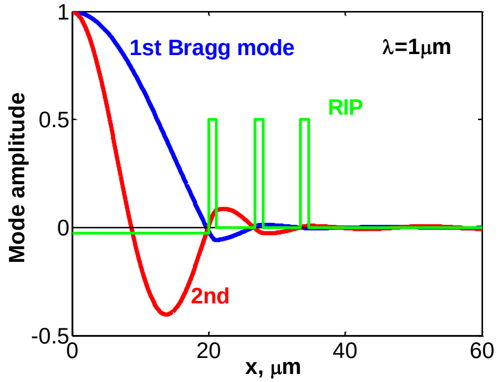
Problems of optimal synthesis

2. Bragg waveguide

The analytic solution obtained above can be used for optimal synthesis of Bragg waveguides

The goal of optimization is to provide fast field attenuation in the waveguide cladding, minimum radiation loss

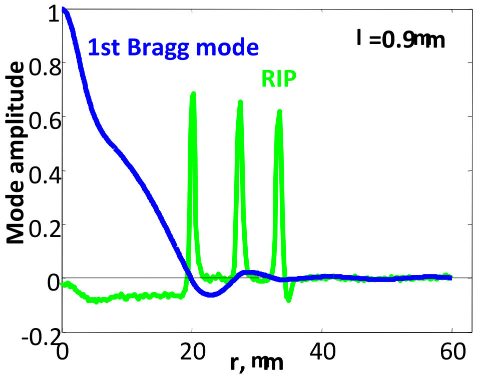




Model index profile and two trapped Bragg modes

Usually, optimal solution is sought by numerical methods, starting from quarter-period layers stack model. Our theory of parametric anti-resonance presents an analytic solution

Realistic Bragg fiber cladding *(P. Roy, M. Likhachev, et al.)* corresponds to an optimal smooth refractive index profile



Problems of optimal synthesis

Numerical example: planar waveguide

Uniform core: n_0

Periodic cladding: $n(x)$

Outer coating: $\bar{n}$

Propagating mode: $$E_y = u(x)\exp(-i\beta z)$$

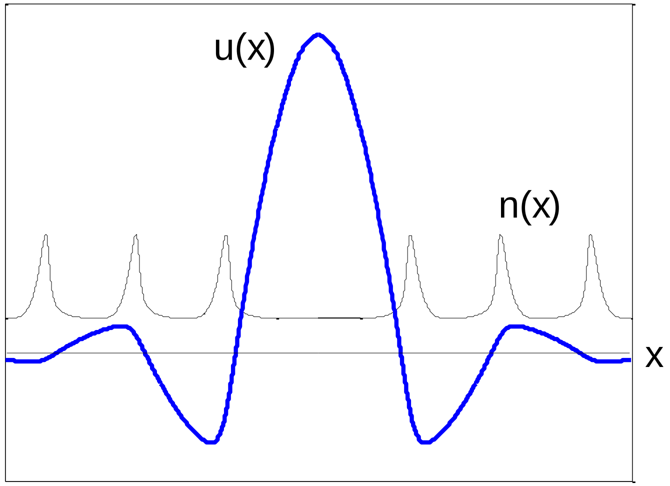
Wave equation: $$u'' + \left[k^2 n^2(x) - \beta^2\right]u = 0$$

Optimization goal: field confinement in the core

Parametric solution: $$q(x) = \sqrt{k^2 n^2(x) - \beta^2} = q_0 \exp\left[\delta\left(\sin 2\psi - \sin 2\psi_0\right)\right]$$

$$x = a + \frac{1}{q_0}\int_{\psi_0}^{\psi} e^{\delta(\sin 2\psi_0 - \sin 2\varphi)}\left(1 - \frac{\delta}{2}\sin 4\varphi\right)d\varphi,$$

$$u(x) = \begin{cases} \cos q_0 x, & x < a \\ -\sin\psi e^{-\frac{\delta}{2}\left[(\psi-\psi_0)+(\sin 2\psi - \sin 2\psi_0)+\frac{1}{4}(\sin 4\psi - \sin 4\psi_0)\right]}, & \\ x > a & \end{cases}$$



Optimal index profile and fundamental Bragg mode

Parametric description of traveling waves

In order to complete the theory we have to learn to describe wave propagation in the transparency zones of a periodic medium and the frequency dependence of the solution. It turns to be a much more difficult problem.

Not going into detailes, the main points of the travelling waves parametric description are as follows. As stated above, in a transparency zone the wave equation $u'' + q^2(x)u = 0$ with periodic $q(x+\Lambda) = q(x)$ has a Floquet solution $u(x) = e^{\mu x}P(x), \quad P(x+\Lambda) = P(x)$ with a purely imaginary coefficient μ

To describe propagating waves in a periodic medium we have to give a definition to the phase of a complex anharmonic wave. By generalization of our previous construction let us define the phase as a homogeneous function of $u(x)$ and $u'(x)$:

$$\boxed{\begin{aligned}\psi(x) &= \frac{1}{2i}\ln\frac{u' + iqu}{u^{*\prime} - iqu^{*}} \\ &\sim \int\left[q + \frac{q'}{2}\,\frac{uu^{*\prime} + u'u^{*}}{|u'|^2 + q^2|u|^2}\right]dx\end{aligned}}$$

For real-valued solutions $u(x)$ this formula reduces to

the familiar expression $\boxed{\psi(x) = \operatorname{arcctg}\left(\dfrac{u'}{qu}\right)}$, whereas

in a smoothly varying medium we obtain WKB phase:

$$\boxed{\begin{aligned}&u \sim q^{-1/2}(x)\exp\left[i\int q(x)dx\right] \\ &\psi(x) = \int q(x)dx\end{aligned}}$$

Solving the wave equation

Our goal is to describe wave propagation in a periodic medium beyond the perturbation theory framework. In order to integrate the wave equation, written in $X(\psi), U(\psi)$ variables, we introduce complex-valued admittance:

$$\boxed{H(\psi) = \frac{u'[x(\psi)]}{u[x(\psi)]} \equiv \frac{\dot{U}(\psi)}{\dot{X}(\psi)U(\psi)}}$$

Amplitude and phase of complex admittance $H(\psi) = Q\rho\exp(i\Phi)$ satisfy a set of nonlinear differential equations:

$$\boxed{\begin{aligned} &\dot{\rho}\cos\Phi + \frac{\rho^2+1}{\rho^2-1}\rho\,\dot{\Phi}\sin\Phi = -\left(\rho^2+1\right) \\ &\dot{\rho}\sin\Phi - \frac{\rho^2+1}{\rho^2-1}\rho\,\dot{\Phi}\cos\Phi = -\rho\frac{\dot{Q}}{Q}\sin\Phi \end{aligned}}$$

Substitution $\rho(\psi) = \sqrt{\dfrac{1+\dot{I}(\psi)}{1-\dot{I}(\psi)}}, \quad Q(\psi) = k\,n_0\exp[g(\psi)]$

leads to a nonlinear second-order equation:

$$\boxed{\ddot{I} + 4I = \dot{g}(\psi)\left(\dot{I}^2 + 4I^2 - 1\right)}$$

A continuum of its solutions can be found for

$$\boxed{\dot{g}(\psi) = M'(I)/M(I)}$$

Solving the wave equation

For arbitrary $\dot{g}(\psi) = M'(I)/M(I)$ we find an energy integral $\dot{I}^2 = 1 - 4I^2 - M^2(I)$

Periodic solutions $I(\psi) = I(\psi + \Theta)$ arise from the integral inversion $\boxed{\psi = \pm\int \frac{dI}{\sqrt{1-4I^2-M^2(I)}}}$

Period $\Theta = 2\int_{I^-}^{I^+} \frac{dI}{\sqrt{1-4I^2-M^2(I)}}$ - integral between turning points $I^{\pm}$

Function $Q(\psi) = k\, n_0 \exp[g(\psi)]$ and parametric solution $\boxed{g(\psi) = \pm\int \frac{M'(I)\,dI}{M(I)\sqrt{1-4I^2-M^2(I)}}}$

are expressed in terms of an arbitrary function $M(I)$:

$$\boxed{\begin{aligned} X(\psi) &= -\int \frac{\dot{H}}{H^2+Q^2}\,d\psi = \int\left(1 - I\,\frac{M'(I)}{M(I)}\right)\frac{d\psi}{Q}, \\ U(\psi) &= U_0 \exp\left(-\int \frac{H\dot{H}}{H^2+Q^2}\,d\psi\right) = U_0 \exp\left[\int\left(1 - I\,\frac{M'}{M}\right)\frac{(2I+iM)}{Q}\,d\psi\right] \end{aligned}}$$

Bloch wave parameters

Complex-valued increment: $\chi + i\nu = \ln\dfrac{U(\psi+\Theta)}{U(\psi)} = 2\displaystyle\int_{\overset{-}{I}}^{\overset{+}{I}} \frac{M(I) - IM'(I)}{4I^2 + M^2(I)}\left(\frac{2I}{M(I)} + i\right) dI$

For even $\boxed{M(I) = M(-I)}$ we get: attenuation $\chi = 0$ (periodic $|U(\psi)|$)

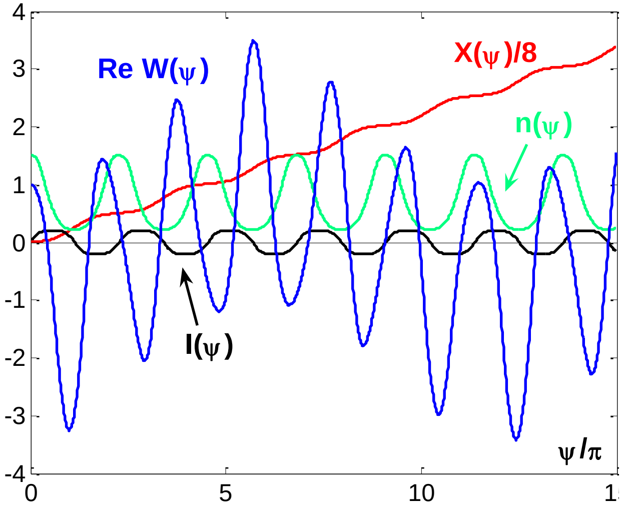
Phase accumulation: $\nu = 4\displaystyle\int_0^{\overset{+}{I}} \frac{M(I) - IM'(I)}{4I^2 + M^2(I)} \frac{dI}{\sqrt{1 - 4I^2 - M^2(I)}}$; modulation period $T = \dfrac{2\pi}{\nu}\Theta$
модуляции:

Numerical example:

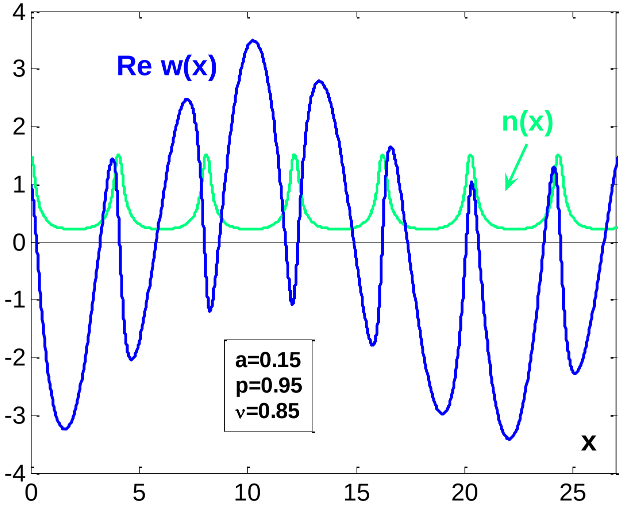
$$\boxed{M(I) = \sqrt{1 - 4a^2 - c^2 I^4}}$$

$$\boxed{\begin{aligned} I(\psi) &= a\sqrt{1+\rho^2}\,\mathrm{sn}\left(\frac{2\psi}{\sqrt{1+\rho^2}}, \rho\right), \\ \rho &= \frac{1 - \sqrt{1 - a^2 c^2}}{ac} \end{aligned}}$$

Parametric resonance:



In physical variables:



Frequency dependence

Consider frequency-dependent solution $\boxed{u(x,\omega)}$ of wave equation $u'' + k^2 n^2(x) u = 0, \quad k = \omega/c$

To round off the construction of the phase parameter method, we should express $\boxed{u(x,\omega)}$ in terms of (ψ, k) variables

Difficulty consists in the necessity to assure frequency-independence of the refraction index profile $n(x)$.
It imposes restrictions upon the variety of model functions $g(\psi,k)$, leading to a nonlinear partial differential equation:

$$X(\psi) = X(\psi,k), \;\; U(\psi) = U(\psi,k)$$

$$N(\psi,k) = \bar{n} \exp\left[g(\psi,k)\right] \equiv n(x)$$

$$\boxed{\left(\frac{kg_k}{\dot{g}}\right)^{\cdot} + 1 = kg_k \cos 2\psi + \frac{1}{2}\dot{g}\sin 2\psi}$$

General solution isn't known; in a narrow frequency band we find an approximate solution:

$$\boxed{g(\psi,k) = g_0\left\{\frac{k_0}{k}\left[\psi + \alpha(k)\right]\right\}, \quad \alpha(k_0) = 0}$$

In a general case, we apply the method of characteristics: look for a contour line:

$$\boxed{\Psi = \Psi(\varphi,k), \quad g\left[\Psi(\varphi,k)\right] = g_0(\varphi)}$$

Simple calculations yield an ODE on a torus:

$$\boxed{\dot{\Psi}(\varphi) = \chi + \frac{1}{2}\dot{g}_0(\varphi)\left(\sin 2\Psi - \chi \sin 2\varphi\right), \quad \chi = \frac{k}{k_0}}$$

Literature

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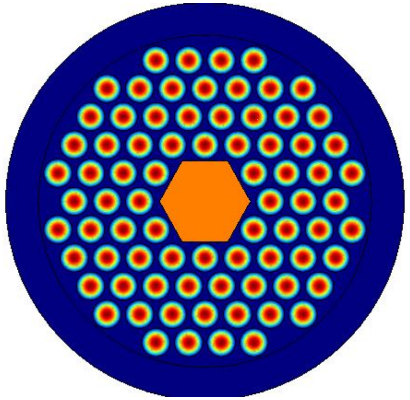
Conclusion

- Applying oscillation phase as an independent variable allows one to obtain new results in the theory of linear and nonlinear oscillations;
- By analogy, we derive an analytical description of electromagnetic wave propagation in non-uniform media;
- Explicit formulas for period and increment of parametric oscillations are useful for solving the problems of optimal synthesis;
- Further development of theory is necessary for efficient description of Bloch waves frequency dependence;
- Generalization to the case of 2D photonic crystals is possible



Thank you for your attention!